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QED vacuum between a conducting and a permeable plate

M V Cougo-Pinto, C Farina, F C Santos and A C Tort

Instituto de Física, Universidade Federal do Rio de Janeiro, CP 68528, Rio de Janeiro, 21945-970 RJ, Brazil

E-mail: marcus@if.ufrj.br and farina@if.ufrj.br and filadelf@if.ufrj.br and tort@if.ufrj.br

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Abstract. We consider the photon field between an unusual configuration of infinite parallel plates, namely: a perfectly conducting plate ($\epsilon \to \infty$) and an infinitely permeable one ($\mu \to \infty$). After quantizing the vector potential in the Coulomb gauge, we obtain explicit expressions for the vacuum expectation values of field operators of the form $\langle E_i E_j \rangle_0$ and $\langle B_i B_j \rangle_0$. These field correlators allow us to re-obtain the Casimir effect for this set-up and to discuss the light velocity shift caused by the presence of plates (Scharnhorst effect: Scharnhorst (1990 *Phys. Lett.* B **236** 354), Barton (1990 *Phys. Lett.* B **237** 559), Barton and Scharnhorst (1993 *J. Phys. A: Math. Gen.* **26** 2037)) for both scalar and spinor QED.

1. Introduction

Ordinary QED deals with processes in unbounded spacetime, with no boundary conditions whatsoever or external fields imposed on, and without compactification of, any spatial dimension. Nonetheless, a number of physical interesting processes involving photons and electrons (bound or not) occur within the confines of physical boundaries, that is, within a cavity. As an example consider the spontaneous emission by an atom. This process is due to the coupling of electromagnetic vacuum oscillations to the bound electron in the atom and in free space is a position-independent observable. However, inside a cavity the vacuum electromagnetic field modes can change substantially and as a consequence the spontaneous emission rate is affected and can become position-dependent [4–6] (see also the textbook by Milonni [7] and references therein). For a half-space 'cavity' comprised by a single metallic wall, for instance, the spontaneous emission rate goes with the reciprocal of the fourth power of the distance of the atom to the wall. In a broader sense, we can say that inside the cavity we can think of the atom as probing the local fluctuations of the electromagnetic vacuum.

The influence of the atom–cavity interaction on the atomic spontaneous emission rate is one among a large number of effects of the so-called cavity QED, a specific branch of QED that basically deals with the influences of the surroundings of a physical system on its radiative properties (see [8, 9] for recent reviews). Although the first cavity QED effect is attributed to Purcell [10], who pointed out that the spontaneous emission process associated with nuclear magnetic moment transitions at radio frequencies could be enhanced if the system were coupled to a ressonant external electric circuit, we can say that the first detailed papers on this subject were those written by Casimir and Polder [11] in which, among other things, forces between polarizable atoms and metallic walls were treated, and by Casimir in his seminal work [12]. In its electromagnetic version, the Casimir effect is the macroscopic attraction force between two neutral, parallel and perfectly conducting infinite surfaces due to the redistribution of normal modes of the vacuum electromagnetic field between them. Experimentally, the Casimir effect between metallic surfaces was first observed by Sparnaay [13] and recently with remarkable accuracy by Lamoreux [14] and Mohideen and Roy [15]. The various Casimir effects have been the subject of many studies, for a review see [16–18].

Still another spectacular instance of cavity QED is the Scharnhorst effect [1, 2]. This effect is basically the velocity shift caused by the change in the zero-point energy density of the quantized electromagnetic field induced by the presence of a perfectly conducting pair of plates. Recall that an external electromagnetic field such as that of propagating light couples to the quantized radiation field through fermionic loops. The Scharnhorst effect is not the only example where nontrivial vacua affect the speed of light. In fact, this subject has attracted the attention of many physicists in recent years [19–24].

It is clear from what was stated above that an analysis of the QED vacuum inside cavities is crucial for an understanding of its observable properties. Here we consider the QED vacuum between an unsual pair of plates. We place an infinite perfectly conducting ($\epsilon \to \infty$) surface parallel to a second infinite perfectly permeable ($\mu \to \infty$) surface held at fixed distance L from the first. This set-up, which we call Boyer plates, was first considered by Boyer in order to compute the corresponding Casimir effect in the framework of random electrodynamics [25] and leads to a repulsive force. This result is somewhat intriguing, since it seems to contradict the explanation given for the usual attractive Casimir effect which suggests that there is a greater number of modes outside the plates than inside. In fact, this is not true: there is only a rearrangement of modes; for a clear explanation of this problem see [26] and references therein. For the generalized ζ-function approach applied to the repulsive Casimir effect for parallel plates geometry see [27, 28]. It is worth mentioning that if we impose on a scalar quantum field mixed boundary conditions (Dirichlet on one plate and Newmann on the other) the resultant Casimir effect will be also repulsive [29]. With respect to the Casimir effect, the problem of quantizing the electromagnetic field between the unusual pair of plates described above and the problem of quantizing a massless scalar field between two planes where mixed boundary conditions are assumed as in [29] are closely connected. In fact, it can be shown that the Casimir energy density for the former case can be computed if in the latter we just multiply the result by a factor of two, to take into account the two possible polarizations of the photon. However, we should emphasize here that this procedure works well only for the plane geometry.

This paper is organized as follows. In section 2 we determine the photon field $\boldsymbol{A}(\boldsymbol{r}, t)$ in the region between Boyer plates making use of the Coulomb gauge. We also evaluate the field operator correlators $\langle \hat{E}_i \hat{E}_j \rangle_0$ and $\langle \hat{B}_i \hat{B}_j \rangle_0$ with the aid of a simple but efficient regularization prescription. In section 3 we apply our results to re-obtain the repulsive Casimir pressure of this set-up. In section 4 we discuss the Scharnhorst effect but for this different situation. In particular, we show that, contrary to the case of the usual pair of conducting plates considered by Scharnhorst [1] and Barton [2], Boyer plates lead to a decrease in the speed of a light for propagation perpendicular to the plates. In section 5 we discuss the Scharnhorst effect for the case of scalar QED trying to keep as much as possible a close analogy with the spinorial QED case. Section 6 contains final remarks and conclusions.

We use natural units so that Planck's constant $\hbar$ and the speed of light c are set equal to one. For the electromagnetic fields we employ the unrationalized Gaussian system. The fine structure constant reads $\alpha = e^2 \approx \frac{1}{137}$.

2. Vacuum electromagnetic field between Boyer's plates

The set-up we consider consists of two infinite parallel surfaces (the plates), one of which is considered to be a perfect conductor ($\epsilon \to \infty$) while the other is supposed to be perfectly permeable ($\mu \to \infty$). We also choose Cartesian axes in such a way that the $\mathcal{OZ}$ axis is perpendicular to both surafces. The perfectly conducting surface is placed at $z = 0$ and the perfectly permeable one at $z = L$. The electromagnetic fields must satisfy the following boundary conditions: (a) the tangential components E_x and E_y of the electric field as well as the normal component B_z of the magnetic field must vanish on the metallic plate at $z = 0$. (b) The tangential components B_x and B_y of the magnetic field must vanish on the permeable plate at $z = L$. It is convenient to work with the vector potential $\boldsymbol{A}(\boldsymbol{r}, t)$ in the Coulomb gauge in which $\boldsymbol{\nabla} \cdot \boldsymbol{A}(\boldsymbol{r}, t) = 0$, $\boldsymbol{E}(\boldsymbol{r}, t) = -\partial \boldsymbol{A}(\boldsymbol{r}, t)/\partial t$ and $\boldsymbol{B}(\boldsymbol{r}, t) = \boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{r}, \boldsymbol{t})$. Then the physical boundary conditions combined with our choice of gauge permit us to translate the boundary conditions in terms of the vector potential components. At $z = 0$ we have:

$$A_x(x, y, 0, t) = 0 \qquad A_y(x, y, 0, t) = 0 \qquad \frac{\partial}{\partial z} A_z(x, y, 0, t) = 0. \tag{1}$$

On the other hand, at $z = L$ we have:

$$\frac{\partial}{\partial z} A_x(x, y, L, t) = 0 \qquad \frac{\partial}{\partial z} A_y(x, y, L, t) = 0 \qquad A_z(x, y, L, t) = 0. \tag{2}$$

The vector potential operator $\boldsymbol{A}(\boldsymbol{r}, t)$ that satisfies the wave equation, the Coulomb gauge condition and the boundary conditions stated above can be written in the form:

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{r}, t) = \frac{1}{\pi} \left(\frac{\pi}{L}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} \int \frac{\mathrm{d}^2\boldsymbol{\kappa}}{\sqrt{\omega}} \Bigg\{ & a^{(1)}(\boldsymbol{\kappa}, n) \hat{\boldsymbol{\kappa}} \times \hat{\boldsymbol{z}} \sin\left[\left(n + \frac{1}{2}\right) \frac{\pi z}{L}\right] \\ & + a^{(2)}(\boldsymbol{\kappa}, n) \left[\hat{\boldsymbol{\kappa}} \frac{\mathrm{i}(n + \frac{1}{2})}{\omega L} \sin\left[\left(n + \frac{1}{2}\right) \frac{\pi z}{L}\right] \right. \\ & \left. - \hat{\boldsymbol{z}} \frac{\kappa}{\omega} \cos\left[\left(n + \frac{1}{2}\right) \frac{\pi z}{L}\right]\right]\Bigg\} \mathrm{e}^{\mathrm{i}(\boldsymbol{\kappa}\cdot\boldsymbol{\rho} - \omega t)} + \text{h.c.} \end{aligned} \tag{3}$$

where $\boldsymbol{\kappa} = (k_x, k_y)$ and $\boldsymbol{\rho}$ is the position vector in the xy-plane. The normal frequencies are given by

$$\omega = \omega(\boldsymbol{\kappa}, n) = \sqrt{\boldsymbol{\kappa}^2 + \left(n + \frac{1}{2}\right)^2 \frac{\pi^2}{L^2}}. \tag{4}$$

The Fourier coefficients $a^{(\lambda)}(\boldsymbol{\kappa}, n)$, where $\lambda = 1, 2$ is the polarization index, are operators acting in the photon state space and satisfy the commutation relation

$$[a^{(\lambda)}(\boldsymbol{\kappa}, n), a^{(\lambda')}(\boldsymbol{\kappa}', n')] = \delta_{\lambda\lambda'} \delta_{nn'} \delta(\boldsymbol{\kappa} - \boldsymbol{\kappa}'). \tag{5}$$

It is convenient to write the vector potential in the general form:

$$\boldsymbol{A}(\boldsymbol{r}, t) = \sum_{n=0}^{\infty} \int \mathrm{d}^2\boldsymbol{\kappa} \sum_{\lambda=1}^{2} a^{(\lambda)}(\boldsymbol{\kappa}, n) \boldsymbol{A}_{\boldsymbol{\kappa}n}^{(\lambda)}(\boldsymbol{r}) \mathrm{e}^{-\mathrm{i}\omega(\boldsymbol{\kappa}, n)t} + \text{h.c.} \tag{6}$$

where $\boldsymbol{A}_{\boldsymbol{\kappa}n}^{(\lambda)}(\boldsymbol{r})$ denotes the mode functions. The mode functions for each polarization state obey the Helmholtz equation and satisfy the boundary conditions stated above. In our case the mode functions are given by:

$$\boldsymbol{A}_{\boldsymbol{\kappa}n}^{(1)}(\boldsymbol{r}) = \frac{1}{\pi} \left(\frac{\pi}{L}\right)^{\frac{1}{2}} \frac{1}{\sqrt{\omega}} \sin\left[\left(n + \frac{1}{2}\right) \frac{\pi z}{L}\right] \mathrm{e}^{-\mathrm{i}\boldsymbol{\kappa}\cdot\boldsymbol{\rho}} \hat{\boldsymbol{\kappa}} \times \hat{\boldsymbol{z}} \tag{7}$$

and

$$\boldsymbol{A}^{(2)}_{\kappa n}(\boldsymbol{r}) = \frac{1}{\pi}\left(\frac{\pi}{L}\right)^{\frac{1}{2}}\frac{1}{\sqrt{\omega}}\left[\hat{\boldsymbol{\kappa}}\frac{\mathrm{i}n\pi}{L\omega}\sin\left[\left(n+\frac{1}{2}\right)\frac{\pi z}{L}\right] - \hat{\boldsymbol{z}}\frac{\kappa}{\omega}\cos\left[\left(n+\frac{1}{2}\right)\frac{\pi z}{L}\right]\right]\mathrm{e}^{-\mathrm{i}\boldsymbol{\kappa}\cdot\boldsymbol{\rho}}. \quad (8)$$

Next we evaluate the electric field operator $\boldsymbol{E}(\boldsymbol{r}, t)$. Recalling that $a^{(\lambda)}(\boldsymbol{\kappa}, n)|0\rangle = 0$ we first write for the correlators $\langle E_i(\boldsymbol{r}, t)E_j(\boldsymbol{r}, t)\rangle_0$ a general expression of the form:

$$\langle E_i(\boldsymbol{r}, t)E_j(\boldsymbol{r}, t)\rangle_0 = \sum_\alpha E_{i\alpha}(\boldsymbol{r})E^*_{j\alpha}(\boldsymbol{r}) \quad (9)$$

where we have introduced the mode functions $E_{i\alpha}(\boldsymbol{r})$ for the electric field. In our case (7) and (8) yield

$$\boldsymbol{E}^{(1)}_{i\kappa n}(\boldsymbol{r}) = \frac{\mathrm{i}}{\pi}\left(\frac{\omega\pi}{L}\right)^{\frac{1}{2}}\sin\left[\left(n+\frac{1}{2}\right)\frac{\pi z}{L}\right]\mathrm{e}^{-\mathrm{i}\boldsymbol{\kappa}\cdot\boldsymbol{\rho}}(\hat{\boldsymbol{\kappa}}\times\hat{\boldsymbol{z}})_i \quad (10)$$

and

$$\boldsymbol{E}^{(2)}_{i\kappa n}(\boldsymbol{r}) = \frac{\mathrm{i}}{\pi}\left(\frac{\omega\pi}{L}\right)^{\frac{1}{2}}\left[\hat{\boldsymbol{\kappa}}_i\frac{\mathrm{i}n\pi}{L\omega}\sin\left[\left(n+\frac{1}{2}\right)\frac{\pi z}{L}\right] - \hat{\boldsymbol{z}}_i\frac{\kappa}{\omega}\cos\left[\left(n+\frac{1}{2}\right)\frac{\pi z}{L}\right]\right]\mathrm{e}^{-\mathrm{i}\boldsymbol{\kappa}\cdot\boldsymbol{\rho}}. \quad (11)$$

Now we substitute (10) and (11) into (9), write $\hat{\boldsymbol{\kappa}}_i = \cos\phi\,\delta_{ix} + \sin\phi\,\delta_{iy}$, $\hat{\boldsymbol{z}}_i = \delta_{iz}$ and $(\hat{\boldsymbol{\kappa}}\times\hat{\boldsymbol{z}})_i = \sin\phi\delta_{ix} - \cos\phi\delta_{iy}$, where ϕ is the azimuthal angle in the xy-plane and compute all angular integrals. In this way we wind up with

$$\begin{aligned}\langle\hat{E}_i(\boldsymbol{r}, t)\hat{E}_j(\boldsymbol{r}, t)\rangle_0 = {}& \left(\frac{2}{\pi}\right)\left(\frac{\pi}{L}\right)\frac{\delta^{\parallel}_{ij}}{2}\sum_{n=0}^{\infty}\sin^2\left[\left(n+\frac{1}{2}\right)\frac{\pi z}{L}\right]\int_0^\infty \mathrm{d}\kappa\,\kappa\omega(\boldsymbol{\kappa}, n)\\ &+ \left(\frac{2}{\pi}\right)\left(\frac{\pi}{L}\right)\left(\frac{\pi}{L}\right)^2\frac{\delta^{\parallel}_{ij}}{2}\sum_{n=0}^{\infty}\sin^2\left[\left(n+\frac{1}{2}\right)\frac{\pi z}{L}\right]\left(n+\frac{1}{2}\right)^2\int_0^\infty \mathrm{d}\kappa\,\kappa\omega^{-1}(\boldsymbol{\kappa}, n)\\ &+ \left(\frac{2}{\pi}\right)\left(\frac{\pi}{L}\right)\delta^{\perp}_{ij}\sum_{n=0}^{\infty}\cos^2\left[\left(n+\frac{1}{2}\right)\frac{\pi z}{L}\right]\int_0^\infty \mathrm{d}\kappa\,\kappa^3\omega^{-1}(\boldsymbol{\kappa}, n) \end{aligned} \quad (12)$$

where $\delta^{\parallel}_{ij} := \delta_{ix}\delta_{jx} + \delta_{iy}\delta_{jy}$ and $\delta^{\perp}_{ij} := \delta_{iz}\delta_{jz}$. Equation (12) gives only a formal expression for the field correlator $\langle E_i(\boldsymbol{r}, t)E_j(\boldsymbol{r}, t)\rangle_0$, since it is an ill-defined expression plagued by divergent terms. Therefore, it lacks physical meaning unless we adopt a regularization prescription. We will first regularize the integrals in equation (12) by using a method based on an analytical extension to the complex plane. Consider for example the first integral that appears on the rhs of (12),

$$\mathcal{I}_1(n, L) := \int_0^\infty \mathrm{d}\kappa\kappa\left(\kappa^2 + \left(n+\frac{1}{2}\right)^2\frac{\pi^2}{L^2}\right)^{1/2}.$$

Since this integral diverges for large κ, it is natural to modify the integrand so that the integral becomes finite. Our choice will be

$$\mathcal{I}_1(n, L) \longrightarrow \mathcal{I}^{reg}_1(n, L; s) := \int_0^\infty \mathrm{d}\kappa\,\kappa\left(\kappa^2 + \left(n+\frac{1}{2}\right)^2\frac{\pi^2}{L^2}\right)^{1/2-s}$$

and after explicit evaluation of this integral we take the limit $s \to 0$. For the moment, let us assume that $\mathrm{Re}\,s$ is large enough to give a precise mathematical meaning for the previous integral. Then, making use of the following integral representation of the Euler beta function, (cf formula 3.251.2 [30]):

$$\int_0^\infty \mathrm{d}x\,x^{\mu-1}(x^2 + a^2)^{\nu-1} = \frac{B}{2}\left(\frac{\mu}{2}, 1-\nu-\frac{\mu}{2}\right)a^{\mu+2\nu-2} \quad (13)$$

where $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$, which holds for Re $(\nu + \frac{\mu}{2}) < 1$ and Re $\mu > 0$, we get

$$\mathcal{I}_1^{reg}(n, L; s) = \frac{1}{2}\left[\left(n + \frac{1}{2}\right)\frac{\pi}{L}\right]^{3-2s}\frac{\Gamma(s - 3/2)}{\Gamma(s - 1/2)} = \frac{1}{(2s - 3)}\left[\left(n + \frac{1}{2}\right)\frac{\pi}{L}\right]^{3-2s}. \tag{14}$$

Inserting this result into the first term of the rhs of (12) (call it $\mathcal{T}_1$), it takes the form:

$$\mathcal{T}_1 = \left(\frac{1}{2s - 3}\right)\left(\frac{\pi}{L}\right)^{3-2s}\frac{\delta_{ij}^{\parallel}}{2L}\left\{\zeta_H\left(2s - 3, \frac{1}{2}\right) - \sum_{n=0}^{\infty}\left(n + \frac{1}{2}\right)^{3-2s}\cos\left[\frac{2(n + \frac{1}{2})\pi z}{L}\right]\right\} \tag{15}$$

where $\zeta_H(z, a)$ is the well known Hurwitz zeta function. Performing the analytical extension to the s-complex plane and taking the limit $s \to 0$, we get

$$\mathcal{T}_1 = -\frac{1}{6\pi}\left(\frac{\pi}{L}\right)^4\delta_{ij}^{\parallel}\left\{\left(-\frac{7}{8}\right)\times\frac{1}{120} - G\left(\frac{\pi z}{L}\right)\right\} \tag{16}$$

where we made use of $\zeta_H(-3, \frac{1}{2}) = (-\frac{7}{8}) \times (\frac{1}{120})$ and defined

$$G(\xi) = -\frac{1}{8}\times\frac{\mathrm{d}^3}{\mathrm{d}\xi^3}\left(\frac{1}{2\sin\xi}\right) = \frac{1}{8}\left(3\frac{\cos^3\xi}{\sin^4\xi} + \frac{5}{2}\frac{\cos\xi}{\sin^2\xi}\right). \tag{17}$$

The other two terms of the rhs of (12) can be treated in a similar way. It is then straightforward to show that

$$\langle E_i(\boldsymbol{r}, t)E_j(\boldsymbol{r}, t)\rangle_0 = \left(\frac{\pi}{L}\right)^4\frac{2}{3\pi}\left[\left(-\frac{7}{8}\right)\left(-\delta^{\parallel} + \delta^{\perp}\right)_{ij}\frac{1}{120} + \delta_{ij}G\left(\frac{\pi z}{L}\right)\right]. \tag{18}$$

Proceeding in the same way as with the evaluation of the electric field correlators, we obtain

$$\langle B_i(\boldsymbol{r}, t)B_j(\boldsymbol{r}, t)\rangle_0 = \left(\frac{\pi}{L}\right)^4\frac{2}{3\pi}\left[\left(-\frac{7}{8}\right)\left(-\delta^{\parallel} + \delta^{\perp}\right)_{ij}\frac{1}{120} - \delta_{ij}G\left(\frac{\pi z}{L}\right)\right] \tag{19}$$

for the magnetic field correlators. A straightforward calculation along the lines given here or the use of time-reversal invariance shows that the correlators $\langle E_i(\boldsymbol{r}, t)B_j(\boldsymbol{r}, t)\rangle_0$ are zero. In passing, observe that no subtractions were required by our regularization procedure. This is a common feature of regularization prescriptions based on analytical extensions. However, other methods in which the subtraction of the field correlators with no boundary conditions are made can also be used yielding the same results.

3. The Casimir effect for Boyer plates

As a first application of the results obtained for the field operator correlators between Boyer plates, let us re-obtain Boyer's result [25] for the Casimir energy density corresponding to this unusual set-up. First, recall that the zero-point energy density ρ_o for the electromagnetic fields is defined by the following vacuum expectation value:

$$\rho_0 = \frac{1}{8\pi}\langle \boldsymbol{E}^2 + \boldsymbol{B}^2\rangle_0. \tag{20}$$

Making use of (18) and (19) we obtain the position-dependent correlators:

$$\langle \boldsymbol{E}^2\rangle_0 = \left(\frac{\pi}{L}\right)^4\frac{2}{3\pi}\left[\frac{7}{8\times 120} + 3G\left(\frac{\pi z}{L}\right)\right] \tag{21}$$

$$\langle \boldsymbol{B}^2\rangle_0 = \left(\frac{\pi}{L}\right)^4\frac{2}{3\pi}\left[\frac{7}{8\times 120} - 3G\left(\frac{\pi z}{L}\right)\right]. \tag{22}$$

If we add these two equations as required by equation (20), the position-dependent terms will cancel out and we obtain

$$\rho_0 = \frac{7}{8}\frac{\pi^2}{720L^4} \tag{23}$$

which is the position-independent Casimir energy density obtained by Boyer [25]. Notice that it is positive and leads to a repulsive force per unit area between the plates [25, 27, 28].

It is also convenient to analyse the behaviour of the correlators $\langle \boldsymbol{E}^2\rangle_0$ and $\langle \boldsymbol{B}^2\rangle_0$ in situations where one of the plates is removed. Let us first consider the limit of a single metal plate located at $z = 0$. This means that we are taking the limit $L \to \infty$ in the previous equations. The results are:

$$\langle \boldsymbol{E}^2\rangle_0 \approx +\frac{3}{4\pi z^4} \tag{24}$$

and

$$\langle \boldsymbol{B}^2\rangle_0 \approx -\frac{3}{4\pi z^4} \tag{25}$$

in agreement with the literature [31]. On the other hand, the limit of a single infinitely permeable plate is obtained by removing the metal plate. This can be accomplished if we consider the limits $L \to \infty$, $z \to \infty$ in the previous results, but with $L - z \ll L$. For this case we obtain:

$$\langle \boldsymbol{E}^2\rangle_0 \approx -\frac{3}{4\pi (z-L)^4} \tag{26}$$

and

$$\langle \boldsymbol{B}^2\rangle_0 \approx +\frac{3}{4\pi (z-L)^4}. \tag{27}$$

Equations (26) and (27) are new results. Let us now turn our attention to one of the most intriguing properties of the QED vacuum between a pair of parallel plates: its anisotropy and the concomitant consequences on the speed of light.

4. The Scharnhorst effect in spinor QED

Basically, the Scharnhorst effect [1, 2] is the velocity shift of a light wave in QED vacuum caused by the presence of two parallel conducting plates for propagation in the region between the plates and in a direction perpendicular to the plates. This effect was shown to occur for small frequencies (soft photon approximation) $\omega \ll m_e$ where m_e is the mass of the electron and in the weak field limit. For the case of metallic plates, Scharnhorst [1], and later Barton [2], showed that the phase velocity, which in this case (small frequencies) coincides with the group velocity, is greater than its value in free space for propagation perpendicular to the plates. However, this does not mean that the signal velocity can be greater than one because in order to determine the wave front velocity the investigation of the dispersion relation in the infinite frequency limit is mandatory. See [3, 32–34] for more detailed discussions on this issue. The Scharnhorst effect with boundary conditions different from the standard ones, namely, those concerning the ubiquitous perfect metallic plates, was also considered [35]. This effect can be understood as follows. The external field which is given by a plane wave propagating in the constrained vacuum interacts with the quantized electromagnetic fields through fermionic loops and therefore any change in the quantized field modes, as for example those caused by imposition of boundary conditions, can in principle modify the wave propagation. In [1, 2] this change was induced by the presence of two perfect parallel conducting plates, while in [35] the same problem was considered for the unusual pair of parallel plates that is being considered here. Since in [1,2] it is assumed that the plates do not impose any type of boundary condition on the fermionic field, the Scharnhorst effect appears only at the two-loop level. Also, because this effect is a perturbative one, it can be obtained by direct computation of the relevant Feynman diagrams that contribute to the effective action: namely, the two possible diagrams for the photon polarization tensor at two-loop level. This was precisely Scharnhorst's approach, who, after using a representation for the photon propagator between two metallic plates obtained by Bordag *et al* [36], found for propagation perpendicular to the plates the result

$$v_{\perp} = 1 + \frac{11\pi^2}{2^2 \times 3^4 \times 5^2} \frac{\alpha^2}{(m_e L)^4}. \tag{28}$$

Later, the same result was rederived by Barton [2] in a more economical way, where the connection with the Casimir energy density is made more explicit. The starting point in Barton's approach is to consider the first corrections to the Maxwell Lagrangian density that originate from the Euler–Heisenberg Lagrangian density [37] after a power expansion for weak fields is made. For fields well below the critical value m_e^2/e, the relevant effective Lagrangian density reads:

$$\begin{aligned} \mathcal{L} &= \mathcal{L}^{(0)} + \mathcal{L}^{(1)} \\ &= \frac{1}{8\pi}(\boldsymbol{E}^2 - \boldsymbol{B}^2) + g[(\boldsymbol{E}^2 - \boldsymbol{B}^2)^2 + 7(\boldsymbol{E}\cdot\boldsymbol{B})^2] \end{aligned} \tag{29}$$

where $g := \alpha^2/5 \times 3^2 \times 2^3 \times \pi^2 m_e^4$. The above Lagrangian density describes the first vacuum polarization effects on slowly varying fields for which the condition $\omega \ll m_e$ holds and is valid only in the weak field approximation. In other words, the first nonlinear corrections to the Maxwell equations originating from QED are described by the quartic terms added to the usual Maxwell Lagrangian density. The corresponding vacuum polarization $\boldsymbol{P}$ and magnetization $\boldsymbol{M}$ are given by:

$$\boldsymbol{P} = \frac{\partial \mathcal{L}^{(1)}}{\partial \boldsymbol{E}} = 4g(\boldsymbol{E}^2 - \boldsymbol{B}^2)\boldsymbol{E} + 14g(\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{B} \tag{30}$$

and

$$\boldsymbol{M} = \frac{\partial \mathcal{L}^{(1)}}{\partial \boldsymbol{B}} = -4g(\boldsymbol{E}^2 - \boldsymbol{B}^2)\boldsymbol{B} + 14g(\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{E}. \tag{31}$$

In order to include a radiative correction into the formalism, we can follow [2] and rewrite the fields in equations (30) and (31) as the sum of two parts, one describing the quantized fields $\boldsymbol{E}_q$ and $\boldsymbol{B}_q$ and the other one describing the classical fields $\boldsymbol{E}_c$ and $\boldsymbol{B}_c$, so that we can write: $\boldsymbol{E} = \boldsymbol{E}_q + \boldsymbol{E}_c$ and $\boldsymbol{B} = \boldsymbol{B}_q + \boldsymbol{B}_c$ and substitute into (30) and (31). This procedure is tantamount to the coupling of the external fields to the quantized ones by means of the intermediary action of a fermionic loop. Keeping only terms which are linear in the classical fields, we obtain the following expressions for the electric susceptibilty $\chi_{ij}^{(e)}$ and magnetic susceptibility $\chi_{ij}^{(m)}$ tensors of the vacuum:

$$\chi_{ij}^{(e)} = 4g[\langle \boldsymbol{E}_q^2 - \boldsymbol{B}_q^2\rangle_0 \delta_{ij} + 2 < E_{qi}E_{qj}\rangle_0] + 14g < B_{qi}B_{qj}\rangle_0 \tag{32}$$

$$\chi_{ij}^{(m)} = 4g[-\langle {\boldsymbol{E}_q}^2 - \boldsymbol{B}_q^2\rangle_0 \delta_{ij} + 2 < B_{qi}B_{qj}\rangle_0] + 14g < E_{qi}E_{qj}\rangle_0. \tag{33}$$

The dieletric and permittivity tensors of the vacuum are:

$$\epsilon_{ij} = \delta_{ij} + 4\pi\chi_{ij}^{(e)} = \delta_{ij} + \Delta\epsilon_{ij} \tag{34}$$

$$\mu_{ij} = \delta_{ij} + 4\pi\chi_{ij}^{(m)} = \delta_{ij} + \Delta\mu_{ij}. \tag{35}$$

The vacuum expectation values in (32) and (33) can be easily calculated with the correlators given by (18) and (19). If we do this, we obtain for $\Delta\epsilon_{ij}$ and $\Delta\mu_{ij}$ the results:

$$\Delta\epsilon_{ij} = g\left(\frac{\pi}{L}\right)^4 \frac{16}{3}\left[\left(-\frac{7}{8}\right)\left(-\delta^{\parallel}+\delta^{\perp}\right)_{ij}\left(\frac{11}{120}\right)+3\delta_{ij}G\left(\frac{\pi z}{L}\right)\right] \tag{36}$$

and

$$\Delta\mu_{ij} = g\left(\frac{\pi}{L}\right)^4 \frac{16}{3}\left[\left(-\frac{7}{8}\right)\left(-\delta^{\parallel}+\delta^{\perp}\right)_{ij}\left(\frac{11}{120}\right)-3\delta_{ij}G\left(\frac{\pi z}{L}\right)\right]. \tag{37}$$

We can also derive single-plate limits for $\Delta\epsilon_{ij}$ and $\Delta\mu_{ij}$. Making use of the approximations to $G(\xi)$ in the limits $\xi \to 0$ and $\xi \to \pi$ we have near the conducting plate at $z = 0$:

$$\Delta\epsilon_{ij} = -\Delta\mu_{ij} = 18g\frac{\delta_{ij}}{z^4} \tag{38}$$

and near the permeable plate at $z = L$:

$$\Delta\epsilon_{ij} = -\Delta\mu_{ij} = -18g\frac{\delta_{ij}}{(z-L)^4}. \tag{39}$$

Now, we are interested in the refraction index $n = \sqrt{\epsilon\mu}$ and its first-order shift:

$$\Delta n = \tfrac{1}{2}(\Delta\epsilon + \Delta\mu) \tag{40}$$

for directions of propagation defined by the Cartesian axes. Let us consider first a plane wave propagating in the $\mathcal{OX}$-direction with the electric field vibrating in the $\mathcal{OZ}$-direction. Then $\Delta\epsilon = \Delta\epsilon_{33}$ and $\Delta\mu = \Delta\mu_{22}$, and from (36), (37) and (40) we can easily verify that $\Delta n = \frac{1}{2}(\Delta\epsilon_{33} + \Delta\mu_{22}) = 0$. We obtain the same result in all instances in which the propagation is parallel to the plane of the plates. As a consequence, the speed of light remains unchanged for propagation parallel to the plates. Now consider a plane wave propagating along the $\mathcal{OZ}$-axis, perpendicularly to the pair of plates. Consider the wave polarized in the $\mathcal{OX}$-direction, for instance. Then $\Delta\epsilon = \Delta\epsilon_{11}$ and $\Delta\mu = \Delta\mu_{22}$, and from (36), (37) and (40) we now obtain:

$$\begin{aligned}\Delta n_{\perp} &\approx \tfrac{1}{2}(\Delta\epsilon_{11} + \Delta\mu_{22})\\ &= +\frac{7}{8}\times\frac{\alpha^2}{(mL)^4}\frac{11\pi^2}{2^2\times 3^4\times 5^2}\end{aligned} \tag{41}$$

which is the result obtained by Scharnhorst [1] and re-obtained by Barton [2] multiplied by the factor $-\frac{7}{8}$. The speed of light in that direction will be:

$$v_{\perp} \approx 1 - \frac{7}{8}\times\frac{\alpha^2}{(mL)^4}\frac{11\pi^2}{2^2\times 3^4\times 5^2} < 1. \tag{42}$$

The velocity of light averaged in direction and polarization for the set-up considered here also satisfies the unifying formula written down by Latorre *et al* [23], which for spinor QED reads

$$\begin{aligned}\langle v\rangle &= \frac{1}{4\pi}\oint v(\theta)\,\mathrm{d}\Omega\\ &= \frac{1}{4\pi}\int_0^{\pi}\left[1-\frac{7}{8}\frac{\alpha^2}{(m_eL)^4}\frac{11\pi^2}{2^2\times 3^4\times 5^2}\cos^2\theta\right]2\pi\sin\theta\,\mathrm{d}\theta\\ &= 1-\frac{44\alpha^2}{135m_e^4}\rho_0\end{aligned} \tag{43}$$

where θ in the last equation is the angle between the direction of the wave propagation and the $\mathcal{OZ}$-direction and ρ_o is given by (23). It can be shown that this formula can be obtained in the weak field limit of a general formalism due to Dittrich and Gies in their analysis of nontrivial vacua [24].

5. The Scharnhorst effect in scalar QED

Quantum electrodynamic phenomena are described by spinor QED as the interaction of charged fermions of spin $\frac{1}{2}$ with the photon field. Although this theory is highly successful, it is also instructive to consider other theories. It may be profitable to study theories that, though not realistic, respect important physical principles as, for instance, the gauge principle and the relativistic invariance principle. This is the case of the so-called scalar QED, which describes charged bosons interacting with the radiation field. Naively, we could think that the interaction between the pseudoscalar charged mesons $\pi^{\pm}$ and $K^{\pm}$ could be described by scalar QED, but this is not true, mainly because these mesons have an inner structure and their interaction is dominated by the strong interaction. In fact, since there are no fundamental charged bosons in nature, scalar QED is of limited application. However, scalar QED can be a useful toy model in many situations and shed some light on interesting physical processes, as we shall see. Without further apologies for considering this model, we consider in this section the Scharnhorst effect in the framework of scalar QED. In the case of scalar QED the analogue of the Euler–Heisenberg effective Lagrangian reads [38]:

$$\mathcal{L}_0^{(1)} = g_0[\tfrac{7}{4}(\boldsymbol{E}^2 - \boldsymbol{B}^2)^2 + (\boldsymbol{E}\cdot\boldsymbol{B})^2] \tag{44}$$

with $g_0 := \alpha^2/5\times 3^2\times 2^5\times \pi^2\times m_o^4$, where m_o is the mass of the hypothetical charged boson associated with one-loop scalar QED. As previously, the polarization $\boldsymbol{P}$ and the magnetization $\boldsymbol{M}$ are defined by equations (30) and (31), and as before we make use of the substitutions $\boldsymbol{E}\to\boldsymbol{E}_q+\boldsymbol{E}_c$ and $\boldsymbol{B}\to\boldsymbol{B}_q+\boldsymbol{B}_c$ and keep only terms linear in the classical fields to obtain the corrections $\Delta\epsilon_{ij}$ and $\Delta\mu_{ij}$ to the dielectric and permittivity tensors of the scalar QED vacuum. The results are

$$\Delta\epsilon_{ij} = 28\pi g_0\langle \boldsymbol{E}^2 - \boldsymbol{B}^2\rangle_0\delta_{ij} + 56\pi g_0\langle E_iE_j\rangle_0 + 8\pi g_0\langle B_iB_j\rangle_0 \tag{45}$$

$$\Delta\mu_{ij} = -28 g_0\langle \boldsymbol{E}^2 - \boldsymbol{B}^2\rangle_0\delta_{ij} + 56\pi g_0\langle B_iB_j\rangle_0 + 8\pi g_0\langle E_iE_j\rangle_0. \tag{46}$$

Now we can make use of these results and analyse the speed of light in a confined scalar QED vacuum. Since the Scharnhorst effect for scalar QED has never been discussed before, we will evaluate the light velocity shifts for two cases, first for two perfectly conducting parallel plates and then for a pair of parallel plates where one is a perfectly conducting plate and the other, an infinitely permeable one.

Consider two perfectly conducting plates, one at $z=0$ and the other at $z=L$. Expressions for the electric and magnetic field correlators for this case can be found in, for instance, [2]. Here we merely state the results

$$\langle E_i(\boldsymbol{r},t)E_j(\boldsymbol{r},t)\rangle_0 = \left(\frac{\pi}{L}\right)^4\frac{2}{3\pi}\left[(-\delta^{\parallel}+\delta^{\perp})_{ij}\ \frac{1}{120} + \delta_{ij}F\left(\frac{\pi z}{L}\right)\right] \tag{47}$$

and

$$\langle B_i(\boldsymbol{r},t)B_j(\boldsymbol{r},t)\rangle_0 = \left(\frac{\pi}{L}\right)^4\frac{2}{3\pi}\left[(-\delta^{\parallel}+\delta^{\perp})_{ij}\ \frac{1}{120} - \delta_{ij}F\left(\frac{\pi z}{L}\right)\right] \tag{48}$$

where $F(\xi)$ is defined by:

$$F(\xi) := -\frac{1}{8}\times\frac{\mathrm{d}^3}{\mathrm{d}\xi^3}\left(\frac{1}{2}\cot\xi\right). \tag{49}$$

Now we take (47) and (48) into (45) and (46) and after some simple manipulations we end up with

$$\Delta\epsilon_{ij} = \frac{16}{3}g_0\left(\frac{\pi}{L}\right)^4\left[(-\delta^{\parallel}+\delta^{\perp})_{ij}\left(\frac{1}{15}\right) + 27\delta_{ij}F(\xi)\right] \tag{50}$$

and

$$\Delta\mu_{ij} = \frac{16}{3} g_0 \left(\frac{\pi}{L}\right)^4 \left[(-\delta^{\parallel} + \delta^{\perp})_{ij} \left(\frac{1}{15}\right) - 27\delta_{ij} F(\xi)\right]. \tag{51}$$

With these results we can now calculate the first correction to the refraction index Δn and, consequently, the correction to the speed of light between two perfectly conducting plates in scalar QED. As in the corresponding case of spinor QED, we find that the speed of light parallel to the plates remains unchanged, but the speed of light perpendicular to the plates is modified by an amount given by

$$\Delta v_{\perp} = -\Delta n = +\frac{16}{45} g_0 \left(\frac{\pi}{L}\right)^4 > 0. \tag{52}$$

It is interesting to compare this result with the analogous effect that takes place in spinor QED. Assuming the same charge for bosons and fermions, we see that the ratio between the light velocity shifts for scalar and usual QED is given by

$$\frac{\Delta v_{\perp}^{\text{scalar}}}{\Delta v_{\perp}^{\text{spinor}}} = \frac{2}{11} \times \left(\frac{m_e}{m_o}\right)^4. \tag{53}$$

Now we repeat the procedure for the unusual pair of plates that we are discussing here. The electric and magnetic field correlators we need are given by equations (18) and (19). Substituting into (45) and (46) we obtain

$$\Delta\epsilon_{ij} = \frac{16}{3} g_0 \left(\frac{\pi}{L}\right)^4 \left[\left(-\frac{7}{8}\right)(-\delta^{\parallel} + \delta^{\perp})_{ij} \left(\frac{1}{15}\right) + 27\delta_{ij} G(\xi)\right] \tag{54}$$

and

$$\Delta\mu_{ij} = \frac{16}{3} g_0 \left(\frac{\pi}{L}\right)^4 \left[\left(-\frac{7}{8}\right)(-\delta^{\parallel} + \delta^{\perp})_{ij} \left(\frac{1}{45}\right) - 27\delta_{ij} G(\xi)\right]. \tag{55}$$

Hence, the speed of light between a metallic plate and an infinitely permeable one in the direction perpendicular to the plates is modified by the amount

$$\Delta v_{\perp} = -\Delta n = -\frac{7}{8} \times \frac{16}{45} g_0 \left(\frac{\pi}{L^4}\right) < 0. \tag{56}$$

The results given by equations (52) and (56) can be unified by considering the velocity of light averaged over all directions of propagation and all possible polarizations. To accomplish this first we write, for instance, for the case of two perfectly conducting plates:

$$v(\theta) = 1 - \frac{16}{45} g_0 \left(\frac{\pi}{L}\right)^4 \cos^2\theta \tag{57}$$

where θ is the angle between the direction of propagation and the $\mathcal{OZ}$-axis. Next we take the average over all possible directions (as before, there is no dependence with the wave polarization). The result is

$$\langle v\rangle = \frac{1}{4\pi} \oint v(\theta)\, \mathrm{d}\Omega = 1 - \frac{8\alpha^2}{135 m_o^4} \left(-\frac{\pi^2}{720 L^4}\right) \tag{58}$$

so that

$$\Delta\langle v\rangle = -\frac{8\alpha^2}{135 m_0^4} \rho_0. \tag{59}$$

Had we used equation (56), which stems from the case of a perfectly conducting plate and an infinitely permeable one, we would have ended up with the same result (59), but this time with ρ_0 given by $\rho_0 = (\frac{7}{8}) \times (\pi^2/720L^4)$. This is the scalar QED version of the unifying formula obtained by Pascual *et al* for spinor QED [23] and it corresponds, as in the spinor QED case, to the weak field limit of a more general approach due to Dittrich and Gies [24].

6. Final remarks

In this paper we have quantized the electromagnetic field between an unusual pair of parallel plates: a perfectly conducting plate and an infinitely permeable one. Then we computed some relevant field operator correlators. With these correlators at our disposal, we easily re-obtained the zero-point energy density for the set-up considered here which was first obtained by Boyer and we also discussed the Scharnhorst effect for this kind of boundary conditions. We also discussed for the first time the Scharnhorst effect in the context of scalar QED and established for this case a unifying formula analogous to that discussed by Latorre *et al* [23] for the usual spinor QED. This unifying formula will also correspond to the weak field limit of Dittrich and Gies' approach if in their formalism the appropriate effective Lagrangian density is used. Moreover, we showed that apart from numerical factors we can say that the usual spinor QED vacuum and the scalar QED vacuum behave in a similar way in the presence of material plates when we assume that these plates impose boundary conditions only on the photon field and not on (fermionic and bosonic) matter fields. However, we expect a different behaviour of the usual QED vacuum and the scalar QED vacuum when we impose boundary conditions on the matter fields, since under the imposition of boundary conditions the former exibits paramagnetic characteristics [39], while the latter exhibits diamagnetic ones [40]. Some work about the influence on the speed of light of boundary conditions imposed on matter fields instead of boundary conditions on the photon fields is in progress, but at the moment the authors are not able to give a conclusive result.

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